Lecture 16

Tuesday, October 18, 2016 9:24 AM

4.4 Indeterminate forms and L'Hospital's Rule

The Second Derivative Test:
Suppose that c is a critical point at which f'{c}=0, that f'{c} exists in a neighborhood of c , and that f'{c} exists. Then f has a relative maximum value at c if f''{c}<0 and a relative minimum value at c if f''{c}>0.If f''{c} =0, the test is not informative.

4.4 Indeterminate Forms & L'Hospital's rule.

Ex $\lim\limits_{x\to 1} \frac{\ln x}{x-1} = ?$ Indeterminate form of type $\frac{0}{0}$.

Plug in $\frac{\ln 1}{1-1} = \frac{0}{0}$

L'Hospital's rule :

Suppose f and g are diff funcs and $g'(x) \neq 0$ near a.

Spse $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$

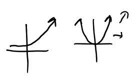
or

$\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$ Ind form of $\frac{\infty}{\infty}$.

Then, $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

Ex $\lim\limits_{x\to 1} \frac{\ln x}{x-1}$ Ind of type $\frac{0}{0}$.

$\stackrel{L'H}{=} \lim\limits_{x\to 1} \frac{1/x}{1} = \lim\limits_{x\to 1} \frac{1}{x} = 1$



Ex $\lim\limits_{x\to \infty} \frac{e^x}{x^2}$ $\quad \frac{\infty}{\infty}$

$\stackrel{L'H}{=} \lim\limits_{x\to\infty} \frac{e^x}{2x}$ $\quad \frac{\infty}{\infty}$

$\stackrel{L'H}{=} \lim\limits_{x\to\infty} \frac{e^x}{2} = \infty$ ✓

Ex $\lim\limits_{x\to 0^+} x\ln x$ $\quad 0 \cdot \pm\infty$

$\lim\limits_{x\to 0^+} x = 0$, $\lim\limits_{x\to 0^+} \ln x = -\infty$

Indeterminate product

We have $\lim\limits_{x\to a} f(x)g(x)$ is of the form $0 \cdot \pm\infty$, we rewrite it as $\lim\limits_{x\to a} \frac{f(x)}{1/g(x)}$

or $\lim\limits_{x\to a} \frac{g(x)}{1/f(x)}$ and use L'H rule.

$0 \cdot \infty = \frac{\infty}{\infty}$

$\frac{0}{0}$

$\frac{-\infty}{\infty}$

$\frac{x}{1/\ln x}$

Ex $\lim\limits_{x\to 0^+} x \cdot \ln x = \lim\limits_{x\to 0^+} \frac{\ln x}{1/x}$ (x^{-1})

$\stackrel{L'H}{=} \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-1x^{-2}} = \lim\limits_{x\to 0^+} \frac{-x^2}{x} = \lim\limits_{x\to 0^+} -x = 0$

Indeterminate Differences :

If $\lim\limits_{x\to a} f(x) = \infty$, $\lim\limits_{x\to a} g(x) = \infty$,

then $\lim\limits_{x\to a} [f(x) - g(x)]$ is called an indeterminate of form $\infty - \infty$.

Ex $\lim\limits_{x\to\infty} x - \ln x = \lim\limits_{x\to\infty} \ln(e^x) - \ln(x)$

$= \lim\limits_{x\to\infty} \ln\left(\frac{e^x}{x}\right)$. Since ln is continuous ,

$$= \underbrace{\lim_{x\to\infty}} \ln\left(\frac{e^x}{x}\right)$$. Since ln is continuous,

$$" = \ln\left(\lim_{x\to\infty} \frac{e^x}{x}\right) \overset{L'H}{=} \ln\left(\lim_{x\to\infty} \frac{e^x}{1}\right) = \infty$$

Indeterminate Powers

$$\lim_{x\to a} f(x)^{g(x)}$$

1) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, Type 0^0

2) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$, Type ∞^0

3) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$, Type $1^{\pm\infty}$

Trick is If $y = f(x)^{g(x)} \Rightarrow \ln y = \ln f(x)^{g(x)} = g(x).\ln f(x)$

and find $\lim_{x\to a} g(x)\ln f(x)$.

Ex $\lim_{x\to 0^+} x^x$ Type 0^0

1) $y = x^x$. Then $\ln y = \ln x^x = x\ln x$

2) $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \underset{(0.\,-\infty)}{x\ln x} = \lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{L'H}{=} 0$ ✓

3) $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} y = \underbrace{\lim_{x\to 0^+}} e^{\ln y} = e^{\lim_{x\to 0^+}\ln y} = e^0 = 1$.

↓ Since e^x is continuous

Ex $\lim_{x\to 1^+} x^{\frac{1}{1-x}}$ $\quad 1^{-\infty}$

1) $y = x^{\frac{1}{1-x}} \Rightarrow \ln y = \ln x^{\frac{1}{1-x}} = \frac{1}{1-x}\ln x$

2) $\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln x}{1-x}$; $\frac{0}{0}$

$$\overset{L'H}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{-1} = \lim_{x\to 1^+} \frac{1}{-x} = -1$$

$\lim_{x\to 1^+} \ln y = -1$

3) $\lim_{x\to 1^+} y = \lim_{x\to 1^+} e^{\ln y} = e^{\lim_{x\to 1^+}\ln y} = e^{-1}$

$y = f^{-1}(x)$

$\Rightarrow y' = \frac{1}{\dots}$

$$\Rightarrow y' = \frac{1}{f'(f^{-1}(x))}$$

$\sqrt{xy} = \sin(1+x^2y)$

$$\frac{1}{2}(xy)^{-1/2}\left[y+xy'\right] = \cos(1+x^2y)\cdot\left[2xy+x^2y'\right]$$

$$\frac{1}{2\sqrt{xy}}y + \frac{xy'}{2\sqrt{xy}} = 2xy\cos(1+x^2y) + x^2y'\cos(1+x^2y)$$

$$\frac{1}{2\sqrt{xy}}y - 2xy\cos(1+x^2y) = x^2y'\cos(1+x^2y) - \frac{xy'}{2\sqrt{xy}}$$

$$\Rightarrow \frac{\frac{y}{2\sqrt{xy}} - 2xy\cos(1+x^2y)}{x^2\cos(1+x^2y) - \frac{x}{2\sqrt{xy}}}$$

$\sin(1+x^2y) = \sqrt{xy}$

~~cos~~ $= \sqrt{1-xy}$

$\cos y \geq 0$

$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\sin(1+x^2y) = \sqrt{xy}$

$\cos(1+x^2y) = ?$

$1+x^2y = \sin^{-1}(\sqrt{xy})$

$\sin(1+x^2y) = \sqrt{xy}$

$\sqrt{1-\sin^2(1+x^2y)}$

$= \sqrt{1-xy}$

26) $y = \dfrac{1-e^{-x}}{1+e^{-x}}$

$\Rightarrow x = \dfrac{1-e^{-y}}{1+e^{-y}}$ STEP 1

$\Rightarrow \quad x = \dfrac{1-e^{-y}}{1+e^{-y}}$ STEP 1

$\Rightarrow \quad x + xe^{-y} = 1 - e^{-y}$ STEP 2

$\Rightarrow \quad xe^{-y} + e^{-y} = 1 - x$

$\Rightarrow \quad e^{-y}(x+1) = 1 - x$

$\Rightarrow \quad e^{-y} = \dfrac{1-x}{1+x}$

$\Rightarrow \quad y = -\ln\left(\dfrac{1-x}{1+x}\right) \Rightarrow f^{-1}(x) = -\ln\left(\dfrac{1-x}{1+x}\right)$ STEP 3